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A remark on the representation theory of the algebra $U_q(sl(n))$ when q is a root of unity

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Abstract. An associative algebra is defined on the set of certain irreducible representation spaces of the algebra $U_q(sl(n))$ when q is a root of unity. This associative algebra is shown to be connected with some algebras defined by the decomposition rule of tensor products of the irreducible representations of certain finite groups.

1. Introduction

In the study of the $\phi_{1,2}$-perturbated minimal models of conformal field theory [1], an interesting phenomenon on the representation theory of the algebra $U_q(sl(2))$ was pointed out. When q is a primitive sixth root of unity, there are only three highest-weight irreducible representations with integer spins. If the corresponding representation spaces are denoted by V_0, V_1, V_2, then a formal decomposition rule of the tensor products of these representations is given as follows:

$$V_{j_1} \otimes V_{j_2} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,4-j_1-j_2)} \oplus V_j \qquad j_1, j_2 = 0, 1, 2. \tag{1.1}$$

It was found that this decomposition rule is precisely the same as that of the irreducible representations of the symmetric group S_3 and, moreover, if the bases of the representation spaces are chosen appropriately, the 6-j symbols of the group S_3 coincide with those of the algebra $U_q(sl(2))$.

Now two questions arise naturally: Is the above-mentioned fact only an incidental phenomenon or is it in some senses general? What is the real nature behind this phenomenon? In this paper we will try to answer positively, to some degree, the first question by considering in general the algebra $U_q(sl(n))$.

We need to introduce some notation. The Hopf algebra $U_q(sl(n))$ is generated by H_i, $X_j^\pm$ ($i, j = 1, 2, \ldots, n-1$) with the following defining relations [2, 3]:

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{H_i/2} - q^{-H_i/2}}{q^{1/2} - q^{-1/2}} \tag{1.2a}$$

$$[X_i^\pm, H_i] = \mp 2X_i^\pm \tag{1.2b}$$

$$[X_i^\pm, H_j] = \pm X_i^\pm \qquad |i-j| = 1 \tag{1.2c}$$

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}_q q^{-k(m-k)/2} (X_i^\pm)^k X_j^\pm (X_i^\pm)^{m-k} = 0 \tag{1.2d}$$

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where $m = 1 - a_{ij}$, and

$$\binom{n}{k}_q = (q^n - 1)(q^{n-1} - 1) \dots (q^{n-k+1} - 1)/(q^k - 1)(q^{k-1} - 1) \dots (q - 1)$$

and the coproduct Δ is defined by

$$\Delta(X_i^\pm) = X_i^\pm \otimes q^{H_i/4} + q^{-H_i/4} \otimes X_i^\pm \tag{1.3a}$$

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i. \tag{1.3b}$$

When q is generic, i.e. it is not a root of unity, an irreducible representation of sl(n) can be deformed to that of $U_q(\mathrm{sl}(n))$, and the representation theory of $U_q(\mathrm{sl}(n))$ does not change from that of sl(n) [4, 5]. So if we denote the fundamental dominant weights of sl(n) by $\{\alpha_i | i = 1, 2, \dots, n-1\}$, then we can denote the irreducible representation of $U_q(\mathrm{sl}(n))$ corresponding to the weight $\alpha = \Sigma_{j=1}^{n-1} m_j\alpha_j$ by $V^{(m_1, m_2, \dots, m_{n-1})}$, where $m_j \geq 0$.

In section 2 we will define an associative algebra on the set of certain irreducible representation spaces of the algebra $U_q(\mathrm{sl}(n))$ when q is a root of unity. In sections 3 and 4, some concrete examples will be calculated so as to investigate more closely the properties of the defined algebra.

2. The definition and some properties of the associative algebra

We will define our algebra by using the algorithm given in [6], in which the generalized Littlewood–Richardson coefficients for the (k, l) representations of the Hecke algebra $H_\infty(q)$ when q is a primitive lth root of unity were calculated. To explain this algorithm, let W denote the affine reflection group on $\mathbb{R}^k$ generated by the reflection

$$\rho: \qquad x \mapsto (x_k + l, x_2, \dots, x_{k-1}, x_1 - l)$$

and the symmetric group S_k. For a Young diagram λ, its length is denoted by $l(\lambda)$. Let n, k, l be positive integers with $1 \leq k < l$. The set of Young diagrams λ with $l(\lambda) \leq k$ and $\lambda_1 - \lambda_k \leq l - k$ is denoted by $\Lambda^{(k,l)}$, and the set of Young diagrams $\lambda \in \Lambda^{(k,l)}$ whose number of boxes is equal to n is denoted by $\Lambda_n^{(k,l)}$.

In [6], $(H^{(k,l)}, \pi^{(k,l)})$ was defined as an approximately finite-dimensional quotient of $H_\infty(q)$ when q is a primitive lth root of unity, $d_{\lambda\mu}^\nu$ were the structure constants of the Littlewood–Richardson ring $\mathscr{H}^{(k,l)} = \bigoplus_n K_0(H_n^{(k,l)})$ of $H^{(k,l)}$ [6]. For each $\lambda \in \Lambda_n^{(k,l)}$, $[p_\lambda]$ denoted the corresponding class of minimal idempotents in $H_n^{(k,l)}$,

$$[p_\lambda] \# [p_\mu] = \sum_{\nu \in \Lambda_{|\lambda|+|\mu|}^{(k,l)}} d_{\lambda\mu}^\nu [p_\nu] \tag{2.1}$$

$d_{\lambda\mu}^\nu$ can then be calculated by

$$d_{\lambda\mu}^\nu = \sum_\eta \varepsilon(w) c_{\lambda\mu}^\eta \tag{2.2}$$

where the summation is over all Young diagrams η for which there exists an element $w \in W$ such that $w(\eta + \delta) = \nu + \delta$; here $\delta = (k-1, k-2, \dots, 0)$ and $\varepsilon(w)$ is the sign of w with $\varepsilon(\rho) = -1$, $c_{\lambda\mu}^\eta$ is the classical Littlewood–Richardson coefficient [7].

When q is generic, the Littlewood–Richardson ring for $H_\infty(q)$ is defined as $\mathscr{H}=\bigoplus_n K_0(H_n)$, and the structure constants of $\mathscr{H}$ are the classical Littlewood–Richardson coefficients $c^\nu_{\lambda\mu}$ [6, 7]:

$$[p_\lambda] \# [p_\mu]=\sum_\eta c^\eta_{\lambda\mu}[p_\eta]. \tag{2.3}$$

We now define the following correspondence between $[p_\lambda]$ and $V^{(m_1,m_2,\ldots,m_{n-1})}$:

$$V^{(m_1,m_2,\ldots,m_{n-1})}\mapsto[p_\lambda]$$

where

$$\lambda=\left(\sum_{j=1}^{n-1} m_j, \sum_{j=2}^{n-1} m_j, \ldots, m_{n-1}\right). \tag{2.4}$$

When $l(\lambda)\leqslant n$ we set

$$[p_\lambda]\mapsto V^{(\bar{m}_1,\bar{m}_2,\ldots,\bar{m}_{n-1})}$$

where

$$m_1=\lambda_1-\lambda_2,\ m_2=\lambda_2-\lambda_3, \ldots, m_{n-1}=\lambda_{n-1}-\lambda_n. \tag{2.5}$$

In the following we will also denote $V^{(m_1,m_2,\ldots,m_{n-1})}$ by $V^{[\lambda]}$ with λ defined by (2.4) and (2.5). Then the decomposition rule of the tensor products of the irreducible representations of $U_q(sl(n))$ when q is generic can be written as [5, 7]

$$V^{[\lambda]}\otimes V^{[\mu]}=\sum_{l(\nu)\leqslant n} c^\nu_{\lambda\mu}V^{[\nu]} \tag{2.6}$$

where $l(\lambda)<n$, $l(\mu)<n$. We observe that (2.6) can be formally obtained by certain restrictions of (2.3) on the length of Young diagrams, so we now define the following algebra for $U_q(sl(n))$ when q is a primitive lth root of unity:

$$Z(n, l)=\{V^{[\lambda]} \mid \lambda\in\Lambda^{(n,l)}, l(\lambda)<n\}$$

with the multiplication on $Z(n, l)$

$$V^{[\lambda]}\otimes V^{[\mu]}=\sum_{\substack{\nu\in\Lambda^{(n,l)}_{|\lambda|+|\mu|}\\ l(\nu)\leqslant n}} \oplus d^\nu_{\lambda\mu}V^{[\nu]} \tag{2.7}$$

where $n\geqslant 3$. From (2.4) and (2.5) we see that $Z(n, l)$ can also be defined as follows:

$$Z(n, l)=\left\{V^{(m_1,m_2,\ldots,m_{n-1})} \,\middle|\, \sum_{j=1}^{n-1} m_j\leqslant -n+l\right\}. \tag{2.8}$$

When q is a primitive lth root of unity the representation of $U_q(sl(n))$ corresponding to $V^{[\lambda]}\in Z(n, l)$ is still irreducible [8].

For $U_q(sl(n))$ the q-dimension [9] is defined by

$$\dim_q V^{(m_1,m_2,\ldots,m_{n-1})}=\frac{\Pi_{j=1}^{n-1}[m_j+1]\,\Pi_{j=1}^{n-2}[m_j+m_{j+1}+2]\ldots[m_1+m_2+\ldots+m_{n-1}+n-1]}{\Pi_{j=2}^{n-1}[j]^{n-j}} \tag{2.9}$$

where

$$[m]=\frac{q^{m/2}-q^{-m/2}}{q^{1/2}-q^{-1/2}}.$$

From our above definitions and [6, 9] we have the following result.

Lemma 2.1. $Z(n, l)$ is an associative algebra with unity $V^{[0]}$, and the following equality holds:

$$\dim_q V^{[\lambda]} \dim_q V^{[\mu]} = \sum_{\substack{\nu \in \Lambda^{(n,l)}_{|\lambda|+|\mu|} \\ l(\nu) \leq n}} d^{\nu}_{\lambda\mu} \dim_q V^{[\nu]} \tag{2.10}$$

where $V^{[\lambda]}$, $V^{[\mu]} \in Z(n, l)$, $n \geq 3$.

Equality (2.10) gives us a hint to find the connections, similar to the one given in the introduction, between $Z(n, l)$ and some finite groups. First, we should find the subalgebras of $Z(n, l)$, which only contain elements with integer q-dimensions.

3. The connection between $Z(n, l)$ and the cyclic groups

Let $C_n = \{g \mid g^n = 1\}$ be the nth order cyclic group. Its n irreducible representations are denoted by $\sigma_j(g) = w_n^j$, where w_n is a primitive nth root of unity. If the representation spaces are denoted by V_n^j, respectively, then we have

$$V_n^i \otimes V_n^j = V_n^k \qquad i, j = 0, 1, \ldots, n-1 \tag{3.1}$$

where $k = i + j \bmod n$. We denote R_n the associative algebra with elements V_n^j $(j = 0, 1, \ldots, n-1)$ and (3.1).

First, we consider the simplest non-trivial algebra $Z(3, 4)$. From section 2 we have

$$Z(3,4) = \{V^{(0,0)}, V^{(1,0)}, V^{(0,1)}\}$$

and

$$\begin{aligned} &V^{(1,0)} \otimes V^{(1,0)} = V^{(0,1)} \\ &V^{(1,0)} \otimes V^{(0,1)} = V^{(0,0)} \\ &V^{(0,1)} \otimes V^{(0,1)} = V^{(1,0)} \\ &V^{(0,0)} \otimes V^{(i,j)} = V^{(i,j)} \qquad i, j = 0, 1. \end{aligned} \tag{3.2}$$

If we set up the correspondence

$$V^{(0,0)} \mapsto V_3^0 \qquad V^{(1,0)} \mapsto V_3^1 \qquad V^{(0,1)} \mapsto V_3^2$$

then obviously $Z(3, 4)$ coincides with R_3 and, moreover, we will show in the appendix that the 6-j symbols of the cyclic group C_3 coincide with that of the algebra $U_q(\mathrm{sl}(3))$ when q is a primitive fourth root of unity.

In general, for any integers n, l with $n \geq 3$, $l > n$, the algebra $Z(n, l)$ contains a subalgebra $H(n, l)$ which is defined as follows:

$$H(n, l) = \{V_{n,l}^j \mid j = 0, 1, \ldots, n-1\}$$

where $V_{n,l}^j = V^{(m_1, m_2, \ldots, m_{n-1})}$ with $m_k = \delta_{k,j} \cdot (l - n)$. It is easy to see that the q-dimension of $V_{n,l}^j$ is equal to 1, so by using the identity (2.10) and our definition of $Z(n, l)$ we have

$$V_{n,l}^{j_1} \otimes V_{n,l}^{j_2} = V_{n,l}^k \tag{3.3}$$

where $k = j_1 + j_2 \bmod n$, $j_1, j_2 = 0, 1, \ldots, n-1$. Now, if we set up the correspondence

$$V_{n,l}^j \mapsto V_n^j \qquad j = 0, 1, \ldots, n-1$$

then the algebra $H(n, l)$ is identified with the algebra R_n related to the cyclic group C_n.

4. The algebras $Z(3,6)$ and $Z(4,6)$

In the following we will sometimes indicate the q-dimension m of $V^{[\lambda]}$ by writing it as $V_m^{[\lambda]}$. We are now interested in the subalgebra of $Z(3,6)$ defined by

$$Q=\{V_1^{(0,0)}, V_3^{(1,1)}, V_1^{(3,0)}, V_1^{(0,3)}\}$$

and with the following formal decomposition rule:

$$\begin{aligned}
&V^{(1,1)}\otimes V^{(1,1)}=V^{(0,0)}\oplus 2V^{(1,1)}\oplus V^{(3,0)}\oplus V^{(0,3)}\\
&V^{(1,1)}\otimes V^{(3,0)}=V^{(1,1)}\\
&V^{(1,1)}\otimes V^{(0,3)}=V^{(1,1)}\\
&V^{(3,0)}\otimes V^{(3,0)}=V^{(0,3)}\\
&V^{(3,0)}\otimes V^{(0,3)}=V^{(0,0)}\\
&V^{(0,3)}\otimes V^{(0,3)}=V^{(3,0)}.
\end{aligned}$$

Consider now the group G of order 12, which is a subgroup of the symmetric group S_4 and is generated by $a=(12)(34)$ and $b=(123)$. The irreducible representations of G are

$$\pi_0(a)=\pi_0(b)=1 \qquad \pi_1(a)=1 \qquad \pi_1(b)=w$$

$$\pi_2(a)=1 \qquad \pi_2(b)=w^2$$

$$\pi_3(a)=\frac{1}{3(w^2-w)}\begin{pmatrix} w-w^2 & 4(1-w) & 4(w^2-1)\\ w^2-1 & w-w^2 & 2(w-1)\\ 1-w & 2(1-w^2) & w-w^2\end{pmatrix}$$

$$\pi_3(b)=\begin{pmatrix} 1 & & \\ & w & \\ & & w^2\end{pmatrix}$$

where $w=e^{2\pi i/3}$. If we denote the corresponding representation spaces also by $V^{(0,0)}$, $V^{(3,0)}$, $V^{(0,3)}$ and $V^{(1,1)}$, respectively, then it is easy to see that the decomposition rule of the tensor products of the above irreducible representations of the finite group G precisely coincide with the formal decomposition rule defined on the subalgebra Q of $Z(3,6)$.

For the algebra $Z(4,6)$ we find the interesting subalgebra

$$Y=\{V_1^{(0,0,0)}, V_1^{(2,0,0)}, V_1^{(0,2,0)}, V_1^{(0,0,2)}, V_2^{(0,1,0)}, V_2^{(1,0,1)}\}$$

with the following formal decomposition rule:

$$\begin{aligned}
&V^{(2,0,0)}\otimes V^{(2,0,0)}=V^{(0,2,0)} && V^{(2,0,0)}\otimes V^{(0,2,0)}=V^{(0,0,2)}\\
&V^{(2,0,0)}\otimes V^{(0,0,2)}=V^{(0,0,0)} && V^{(0,2,0)}\otimes V^{(0,2,0)}=V^{(0,0,0)}\\
&V^{(0,2,0)}\otimes V^{(0,0,2)}=V^{(2,0,0)} && V^{(0,0,2)}\otimes V^{(0,0,2)}=V^{(0,2,0)}\\
&V^{(0,1,0)}\otimes V^{(2,0,0)}=V^{(1,0,1)} && V^{(0,1,0)}\otimes V^{(0,2,0)}=V^{(0,1,0)}\\
&V^{(0,1,0)}\otimes V^{(0,0,2)}=V^{(1,0,1)} && V^{(1,0,1)}\otimes V^{(2,0,0)}=V^{(0,1,0)}\\
&V^{(1,0,1)}\otimes V^{(0,2,0)}=V^{(1,0,1)} && V^{(1,0,1)}\otimes V^{(0,0,2)}=V^{(0,1,0)}\\
&V^{(0,1,0)}\otimes V^{(0,1,0)}=V^{(0,0,0)}\oplus V^{(1,0,1)}\oplus V^{(0,2,0)}\\
&V^{(0,1,0)}\otimes V^{(1,0,1)}=V^{(2,0,0)}\oplus V^{(0,1,0)}\oplus V^{(0,0,2)}\\
&V^{(1,0,1)}\otimes V^{(1,0,1)}=V^{(0,0,0)}\oplus V^{(1,0,1)}\oplus V^{(0,2,0)}.
\end{aligned}$$

We now consider the finite group D of order 12, with generators a, b which satisfy the following relations:

$$a^3 = 1 \qquad b^4 = 1 \qquad ab = ba^{-1}.$$

This finite group has the following six irreducible representations:

$$\sigma_0(a) = 1 \qquad \sigma_0(b) = 1 \qquad \sigma_1(a) = 1 \qquad \sigma_1(b) = i$$

$$\sigma_2(a) = 1 \qquad \sigma_2(b) = -1 \qquad \sigma_3(a) = 1 \qquad \sigma_3(b) = -i$$

$$\sigma_4(a) = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \qquad \sigma_4(b) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\sigma_5(a) = \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix} \qquad \sigma_5(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $w = e^{2\pi i/3}$. If we denote the corresponding representation spaces also by $V^{(0,0,0)}$, $V^{(2,0,0)}$, $V^{(0,2,0)}$, $V^{(0,0,2)}$, $V^{(0,1,0)}$ and $V^{(1,0,1)}$, respectively, as in the case of $Z(3, 6)$ then the decomposition rule of the tensor products of the above irreducible representations of the finite group D coincides with the formal decomposition rule defined on the subalgebra Y.

5. Conclusion

We have shown the connection between the decomposition rule of the tensor products of the irreducible representations of the algebra $U_q(sl(n))$ and that of some finite groups. However, we have not yet discussed the connections between their 6-j symbols, which is also very interesting. To illustrate this connection we will show in detail in the appendix that the 6-j symbols of the cyclic group C_3 coincide with that of the algebra $U_q(sl(3))$. However, since to calculate the 6-j symbols of the algebra $U_q(sl(n))$ is difficult, in general, we can now only give the following conjecture.

Conjecture. Under our correspondence between the irreducible representations of the finite groups C_n, G, D and that of the algebra $U_q(sl(n))$, the 6-j symbols of these finite groups coincide with that of the algebra $U_q(sl(n))$.

We hope that our results will also have applications in conformal field theory [1].

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Appendix

We will calculate in detail some of the 6-j symbols of the algebra $U_q(sl(3))$, which will be connected with the 6-j symbols of the cyclic group C_3. We first introduce the notation of 6-j symbols [10].

Let $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ be the tensor product of three irreducible representations of a given algebra A. There are two simple ways to obtain irreducible components in this representation. One is to decompose first $V^{j_1}\otimes V^{j_2}=\Sigma_{j_{12}}\oplus V^{j_{12}}$, and then to take irreducible submodules in $V^{j_{12}}\otimes V^{j_3}$. In this way we obtain a complete orthogonal base in $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$. Denote the base elements of a submodule V^j in $V^{j_1}\otimes V^{j_2}\otimes V^{j_3}$ by $e_m^{j_{12}j}(j_1j_2|j_3)$, where $m=1,2,\ldots,\dim V^j$. The second way is to decompose first $V^{j_2}\otimes V^{j_3}=\Sigma_{j_{23}}\oplus V^{j_{23}}$, and then $V^{j_1}\otimes V^{j_{23}}$. In this way we obtain $e_m^{j_{23}j}(j_1|j_2j_3)$ ($m=1,2,\ldots,\dim V^j$), which are base elements of the irreducible submodule V^j.

The elements of the matrix which connects the above two bases are called 6-j symbols:

$$e_m^{j_{12}j}(j_1j_2|j_3)=\sum_{j_{23}}\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{Bmatrix} e_m^{j_{23}j}(j_1|j_2j_3)$$

We will denote the elements of an orthonormal base of an irreducible submodule $V^{j_{12}}$ in $V^{j_1}\otimes V^{j_2}$ by $e_m^{j_{12}}(j_1,j_2)$, ($m=1,2,\ldots,\dim V^{j_{12}}$).

Now consider the irreducible representations of $U_q(sl(3))$. We will denote the representation spaces $V^{(0,0)}$, $V^{(1,0)}$, $V^{(0,1)}$, $V^{(2,0)}$, $V^{(0,2)}$ and $V^{(1,1)}$ by V^j ($j=0,1,\ldots,5$), respectively, and the corresponding representations by ρ^j ($j=0,1,\ldots,5$). Denote the base elements of V^j by e_k^j ($k=1,2,\ldots,\dim V^j$); the matrix $E_{i,j}=(a_{k,l})$ is defined by $a_{k,l}=\delta_{ik}\delta_{jl}$. Then we have

$$\rho^0(X_i^\pm)=\rho^0(H_i)=0 \qquad i=1,2$$

$$\rho^1(X_1^+)=E_{2,1} \qquad \rho^1(X_2^+)=E_{3,2} \qquad \rho^1(X_1^-)=E_{1,2}$$

$$\rho^1(X_2^-)=E_{2,3} \qquad \rho^1(H_1)=E_{1,1}-E_{2,2} \qquad \rho^1(H_2)=E_{2,2}-E_{3,3}$$

$$\rho^2(X_1^\pm)=\rho^1(X_2^\pm) \qquad \rho^2(X_2^\pm)=\rho^1(X_1^\pm)$$

$$\rho^2(H_1)=\rho^1(H_2) \qquad \rho^2(H_2)=\rho^1(H_1)$$

$$\rho^3(X_1^+)=[2]^{1/2}E_{2,1}+[2]^{1/2}E_{4,2}+E_{5,3}$$

$$\rho^3(X_2^+)=E_{3,2}+[2]^{1/2}E_{5,4}+[2]^{1/2}E_{6,5}$$

$$\rho^3(X_1^-)=[2]^{1/2}E_{1,2}+[2]^{1/2}E_{2,4}+E_{3,5}$$

$$\rho^3(X_2^-)=E_{2,3}+[2]^{1/2}E_{4,5}+[2]^{1/2}E_{5,6}$$

$$\rho^3(H_1)=2E_{1,1}+E_{3,3}-2E_{4,4}-E_{5,5}$$

$$\rho^3(H_2)=E_{2,2}-E_{3,3}+2E_{4,4}-2E_{6,6}$$

$$\rho^4(X_1^\pm)=\rho^3(X_2^\pm) \qquad \rho^4(X_2^\pm)=\rho^3(X_1^\pm)$$

$$\rho^4(H_1)=\rho^3(H_2) \qquad \rho^4(H_2)=\rho^3(H_1)$$

$$\rho^5(X_1^+)=E_{2,1}+(c_1[2]^{-1/2})+d_1[2]^{1/2}E_{4,3}+(c_2[2]^{-1/2}+d_2[2]^{1/2})E_{5,3}$$
$$+[2]^{1/2}\left(\frac{c_1}{t}E_{6,5}-\frac{c_2}{t}E_{6,4}\right)+E_{8,7}$$

$$\rho^5(X_2^+) = E_{3,1} + (c_1[2]^{1/2} + d_1[2]^{-1/2})E_{4,2} + (c_2[2]^{1/2} + d_2[2]^{-1/2})E_{5,2} + [2]^{1/2}\left(\frac{d_2}{t}E_{7,4} - \frac{d_1}{t}E_{7,5}\right) + E_{8,6}$$

$$\rho^5(X_1^-) = E_{1,2} + [2]^{1/2}\left(\frac{c_1}{t}E_{3,5} - \frac{c_2}{t}E_{3,4}\right) + (c_1[2]^{-1/2} + d_1[2]^{1/2})E_{4,6} + (c_2[2]^{-1/2} + d_2[2]^{1/2})E_{5,6} + E_{7,8}$$

$$\rho^5(X_2^-) = E_{1,3} + [2]^{1/2}\left(\frac{d_2}{t}E_{2,4} - \frac{d_1}{t}E_{2,5}\right) + ([2]^{1/2}c_1 + [2]^{-1/2}d_1)E_{4,7} + ([2]^{1/2}c_2 + [2]^{-1/2}d_2)E_{5,7} + E_{6,8}$$

$$\rho^5(H_1) = E_{1,1} - E_{2,2} + 2E_{3,3} - 2E_{6,6} + E_{7,7} - E_{8,8}$$

$$\rho^5(H_2) = E_{1,1} + 2E_{2,2} - E_{3,3} + E_{6,6} - 2E_{7,7} - E_{8,8}$$

where $t = c_1d_2 - c_2d_1$, c_i, d_i are q-numbers which satisfy $t \neq 0$.

It is now easy to obtain

$$e_1^0(1,2) = -[3]^{-1/2}(q^{-1/2}e_3^1 \otimes e_1^2 - e_2^1 \otimes e_2^2 + q^{1/2}e_1^1 \otimes e_3^2)$$

$$e_1^4(2,2) = e_1^2 \otimes e_1^2 \qquad e_2^4(2,2) = [2]^{-1/2}(q^{1/4}e_2^2 \otimes e_1^2 + q^{-1/4}e_1^2 \otimes e_2^2)$$

$$e_3^4(2,2) = [2]^{-1/2}(q^{1/4}e_3^2 \otimes e_1^2 + q^{-1/4}e_1^2 \otimes e_3^2)$$

$$e_4^4(2,2) = e_2^2 \otimes e_2^2 \qquad e_6^4(2,2) = e_3^2 \otimes e_3^2$$

$$e_5^4(2,2) = [2]^{-1/2}(q^{1/4}e_3^2 \otimes e_2^2 + q^{-1/4}e_2^2 \otimes e_3^2)$$

$$e_1^1(2,2) = [2]^{-1/2}(q^{1/4}e_1^2 \otimes e_2^2 - q^{-1/4}e_2^2 \otimes e_1^2)$$

$$e_2^1(2,2) = [2]^{-1/2}(q^{1/4}e_1^2 \otimes e_3^2 - q^{-1/4}e_3^2 \otimes e_1^2)$$

$$e_3^1(2,2) = [2]^{-1/2}(q^{1/4}e_2^2 \otimes e_3^2 - q^{-1/4}e_3^2 \otimes e_2^2)$$

$$e_1^2(1,1) = [2]^{-1/2}(q^{1/4}e_1^1 \otimes e_2^1 - q^{-1/4}e_2^1 \otimes e_1^1)$$

$$e_2^2(1,1) = [2]^{-1/2}(q^{1/4}e_1^1 \otimes e_3^1 - q^{-1/4}e_3^1 \otimes e_1^1)$$

$$e_3^2(1,1) = [2]^{-1/2}(q^{1/4}e_2^1 \otimes e_3^1 - q^{-1/4}e_3^1 \otimes e_2^1)$$

$$e_1^2(1,4) = [2]^{-1/2}q^{-1/4}e_3^1 \otimes e_1^4 - [2]^{-1}e_2^1 \otimes e_2^4 + [2]^{-1}q^{1/2}e_1^1 \otimes e_3^4$$

$$e_2^2(1,4) = q^{-1/2}e_3^1 \otimes e_2^4 - [2]^{-1/2}q^{-1/4}e_2^1 \otimes e_4^4 - [2]^{-1}e_3^1 \otimes e_2^4 + [2]^{-1}q^{1/2}e_1^1 \otimes e_5^4$$

$$e_3^2(1,4) = [2]^{-1}q^{-1}e_3^1 \otimes e_1^4 - [2]^{-1}q^{-1/2}e_2^1 \otimes e_5^4 + [2]^{-1/2}q^{1/4}e_1^1 \otimes e_6^4.$$

So we have

$$e_1^{0,2}(1,2|2) = -[3]^{-1/2}(q^{-1/2}e_3^1 \otimes e_1^2 - e_2^1 \otimes e_2^2 + q^{1/2}e_1^1 \otimes e_3^2) \otimes e_1^2$$

$$e_1^{1,2}(1|2,2) = [2]^{-1/2}\{q^{1/4}[2]^{-1/2}e_1^1 \otimes (q^{1/4}e_1^2 \otimes e_3^2 - q^{-1/4}e_3^2 \otimes e_1^2) - q^{-1/4}[2]^{-1/2}e_2^1 \otimes (q^{1/4}e_1^2 \otimes e_2^2 - q^{-1/4}e_2^2 \otimes e_1^2)\}$$

$$e_1^{4,2}(1|2,2) = [2]^{-1/2}q^{-3/4}e_3^1 \otimes e_1^2 \otimes e_1^2 - [2]^{-3/2}e_2^1 \otimes (q^{1/4}e_2^2 \otimes e_1^2 + q^{-1/4}e_1^2 \otimes e_2^2) + [2]^{-3/2}q^{1/2}e_1^1 \otimes (q^{1/4}e_3^2 \otimes e_1^2 + q^{-1/4}e_1^2 \otimes e_3^2).$$

It is easy to see that

$$e_1^{0,2}(1,2|2)=[3]^{-1/2}e_1^{1,2}(1|2,2)-[2]^{1/2}[3]^{-1/2}q^{1/4}e_1^{4,2}(1|2,2)$$

and, similarly, we have

$$e_m^{0,2}(1,2|2)=[3]^{-1/2}e_m^{1,2}(1|2,2)-[2]^{1/2}[3]^{-1/2}q^{1/4}e_m^{4,2}(1|2,2) \qquad m=2,3.$$

So we obtain

$$\begin{Bmatrix}1 & 2 & 0\\ 2 & 2 & 1\end{Bmatrix}=[3]^{-1/2}.$$

In a similar way we obtain

$$\begin{Bmatrix}1 & 1 & 2\\ 2 & 1 & 0\end{Bmatrix}=[3]^{-1/2} \qquad \begin{Bmatrix}1 & 2 & 0\\ 1 & 1 & 0\end{Bmatrix}=[3]^{-1}$$

$$\begin{Bmatrix}2 & 1 & 0\\ 2 & 2 & 0\end{Bmatrix}=[3]^{-1} \qquad \begin{Bmatrix}2 & 1 & 0\\ 1 & 1 & 2\end{Bmatrix}=[3]^{-1/2}$$

$$\begin{Bmatrix}2 & 2 & 1\\ 1 & 2 & 0\end{Bmatrix}=[3]^{-1/2}.$$

When q is a primitive fourth root of unity, all the above 6-j symbols are equal to 1 so, if we identify the representation spaces V_3^j of the cyclic group C_3 with V^j ($j=0,1,2$), then we see that the 6-j symbols of the cyclic group C_3 coincide with those of the algebra $U_q(sl(3))$.

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